

# LOWER BOUNDARIES FOR PARAMETRIC ESTIMATIONS IN DIFFERENT NORMS

E.Ostrovsky<sup>a</sup>, L.Sirota<sup>b</sup>

<sup>a</sup> Corresponding Author. Department of Mathematics and computer science, Bar-Ilan University, 84105, Ramat Gan, Israel. E-mail: eugostrovsky@list.ru

<sup>b</sup> Department of Mathematics and computer science. Bar-Ilan University, 84105, Ramat Gan, Israel. E-mail: sirota3@bezeqint.net

## Abstract.

We establish some new *non-asymptotical* lower bounds for deviation of regular unbiased estimation of unknown parameter from its true value in different norms, alike the classical Rao - Kramer's inequality.

We show that if the new norm is weaker than ordinary Hilbertian norm, that the rate of convergence of arbitrary regular unbiased estimate does not exceed  $1/\sqrt{n}$ , and if the new norm is stronger than one, the rate of convergence of the well-known Maximal Likelihood Estimate (MLE) is also equal to  $1/\sqrt{n}$ .

*Key words and phrases.* Probability, estimate, bias, density of distribution, estimate and unbiased regular estimate, likelihood function and estimation, Rao-Kramer's inequality, Rosenthal constants and inequality; ordinary, strong and weak normal rearrangement invariant space, its conjugate (dual) and associate space, Lebesgue-Riesz, Orlicz, CLT norm, Grand Lebesgue and Lorentz spaces and norms, moment, random variable and random vector (r.v.), sample, Fisher's information and its generalization on the arbitrary rearrangement invariant space.

## 1 Statement of problem. Notations. Assumptions.

Let  $(\Omega, B, \mathbf{P})$  be non - trivial probability space with expectation  $\mathbf{E}$  and variance  $\text{Var}$ ,  $(X = \{x\}, \mathcal{A}, \mu)$  be measurable space equipped with sigma - finite measure  $\mu$ ,  $\Theta = \{\theta\}$  be connected subset of real line, i.e. open, semi-open or closed interval,  $\theta_0$  be a fixed interior point in the set  $\Theta$ .

It is sufficient to suppose for example building that on the set  $\Omega$  there exists an uniform distributed random variable.

Let also  $f = f(x, \theta)$ ,  $x \in X$ ,  $\theta \in \Theta$  be differentiable relative the parameter  $\theta$  strictly positive density, i.e. numerical measurable normed function:

$$\int_X f(x, \theta) \mu(dx) = 1, \quad \forall x \in X, \quad \forall \theta \in \Theta \Rightarrow f(x, \theta) > 0. \quad (1.1)$$

We suppose that the random variable  $\xi : \Omega \rightarrow R$  has a density  $f(x, \theta_0)$  :

$$\mathbf{P}(\xi \in G) = \int_G f(x, \theta_0) \mu(dx),$$

i.e. the value  $\theta_0$  is true value of the parameter  $\theta$ .

Let us denote by  $L(\xi) = L(\xi, \theta)$  the ordinary likelihood function:

$$L(\xi) = L(\xi, \theta) \stackrel{\text{def}}{=} \log f(\xi, \theta).$$

The following function

$$\theta \rightarrow L(\xi, \theta) - L(\xi, \theta_0) = \log[f(\xi, \theta)/f(\xi, \theta_0)]$$

is named a *contrast* function.

The r.v.  $\xi$  may be also a vector with values in the space  $R^n$ , in particular, may be a sample of a volume  $n$  :

$$\xi = \vec{\xi} = \{\eta(1), \eta(2), \dots, \eta(n)\}, \quad n = 2, 3, \dots;$$

where the r.v.  $\{\eta(i)\}$  are i. i.d. with the positive density which we denote by  $g(x, \theta)$ . Of course,

$$f(x, \theta) = \prod_{i=1}^n g(x_i, \theta), \quad x = \{x(1), x(2), \dots, x(n)\}.$$

We denote also in this case

$$l_i = l_i(\eta(i)) = l_i(\eta(i), \theta) = \partial \log g(\eta(i), \theta) / \partial \theta; \quad l = l_1.$$

Further, let  $\hat{\theta} = \hat{\theta}(\xi)$  be some unbiased *regular* in the sense of the monograph [6] estimate of the parameter  $\theta$  :

$$\mathbf{E} \hat{\theta} = \int_X \hat{\theta}(x) f(x, \theta) \mu(dx) = \theta, \quad \theta \in \Theta, \quad (1.2)$$

All we need is the following two equalities:

$$\mathbf{E} \left[ (\hat{\theta} - \theta_0) \cdot \frac{\partial \log f(\xi, \theta)}{\partial \theta} \right] = \int_X [(\hat{\theta}(x) - \theta_0) \cdot \partial \log f(x, \theta) / \partial \theta] \mu(dx) = 1; \quad (1.3)$$

$$\mathbf{E} \frac{\partial \log f(\xi, \theta)}{\partial \theta} = \int_X [\partial \log f(x, \theta) / \partial \theta] \mu(dx) = 0. \quad (1.4)$$

**Our purpose in this report is obtaining the lower non - asymptotical estimation of Rao-Kramer's type for the deviation  $\sqrt{n}||(\hat{\theta}_n - \theta_0)||Z$ , i.e. under the classical norming sequence  $\sqrt{n}$ , for some different r.i. norms over source probability space  $|| \cdot ||Z$ .**

The *upper* non - asymptotical estimations for these deviation, and as a consequence an exponentially exact confidential interval for the unknown parameter  $\theta_0$ , under modern terms: majorizing measures, generic chaining etc. for the MLE estimates was derived in the article [10]; see also [2], chapter 2, section 23; [8], chapter 3, Lemma 3.19.

It is clear that the norm  $\|\cdot\|_Y$  should be substantially weaker as the classical  $L_2(\Omega, \mathbf{P})$ , as well as for the investigation of upper estimation this norm should be stronger as one.

Note briefly that the case of biased estimate, multivariate parameter and both this circumnutations may be investigated quite analogously.

## 2 General estimates

Let  $(Y, \|\cdot\|_Y)$  be arbitrary rearrangement invariant (r.i.) space over  $(\Omega, B, \mathbf{P})$ . Reader can found using for us facts about the theory of this spaces in the classical monograph [1].

*We accept that all considered in this article r.i. spaces will be constructed over our probability space  $(\Omega, B, \mathbf{P})$ .*

We denote as ordinary by  $(Y', \|\cdot\|_{Y'})$  the associate space relative the "scalar product"

$$(\zeta, \tau) = \mathbf{E}\zeta\tau = \int_{\Omega} \zeta(\omega) \tau(\omega) \mathbf{P}(d\omega), \quad (2.1)$$

so that  $(Y', \|\cdot\|_{Y'})$  is again r.i. space and

$$\|\tau\|_{Y'} = \sup_{\zeta: \|\zeta\|_Y=1} \left[ \frac{|(\zeta, \tau)|}{\|\zeta\|_Y} \right]; \quad \|\zeta\|_Y = \sup_{\tau: \|\tau\|_{Y'}=1} \left[ \frac{|(\zeta, \tau)|}{\|\tau\|_{Y'}} \right]. \quad (2.2)$$

Therefore,

$$|(\zeta, \tau)| \leq \|\zeta\|_Y \cdot \|\tau\|_{Y'}, \quad (2.3)$$

the generalized Hölder's inequality.

Let  $\eta$  be any centered r.v. belonging to the r.i. space  $Y$ . We define (and denote) analogously M.Ledoux and M.Talagrand [9], p.274 - 275 the following so-called  $CLT(Y) = C(Y)$  norm for  $\eta$  as follows:

$$\|\eta\|_{CLT(Y)} = \|\eta\|_{C(Y)} \stackrel{def}{=} \sup_n \|n^{-1/2} \sum_{i=1}^n \eta_i\|_Y, \quad (2.4)$$

where  $\{\eta_i\}$  are independent copies of  $\eta$ . It will be presumed without loss of generality that the probability space is sufficiently rich, see beginning of this article.

Obviously, if  $\|\eta\|_{C(Y)} < \infty$  then  $\|\eta\|_{L_2(\Omega, P)} < \infty$ .

Note but that our definition does not coincides with the definition of M.Ledoux and M.Talagrand.

**Theorem 2.1.** Let  $\xi = \{\eta(i)\}$ ,  $i = 1, 2, \dots, n$  be a sample of a volume  $n$ . Suppose that there exists a r.i. space  $(Y, \|\cdot\|_Y)$  over our probability space such that the r.v.  $l = l(\eta_1, \theta)$  belongs to the space  $CLT(Y) : 0 < \|l\|_{CLT(Y)} < \infty$ . Then for arbitrary unbiased regular estimate  $\hat{\theta}$

$$\sqrt{n} \|\hat{\theta} - \theta_0\|_{Y'} \geq \frac{1}{\|l\|_{CLT(Y)}}. \quad (2.5)$$

**Proof.** We start from the relations (1.3) and (1.4):

$$1 = (\hat{\theta} - \theta_0, \sum_{i=1}^n l_i). \quad (2.6)$$

We use the (generalized) Hölder's inequality (2.3):

$$1 \geq \| \hat{\theta} - \theta_0 \|_{Y'} \cdot \left\| \sum_{i=1}^n l_i \right\|_Y = \sqrt{n} \cdot \| \hat{\theta} - \theta_0 \|_{Y'} \cdot \| n^{-1/2} \sum_{i=1}^n l_i \|_Y.$$

It follows from the direct definition of the  $CLT(Y)$  norm

$$1 \geq \sqrt{n} \cdot \| \hat{\theta} - \theta_0 \|_{Y'} \cdot \| l \|_{CLT(Y)},$$

which is equivalent to the assertion (2.5) of theorem 2.1.

If for example  $Y = Y' = L_2(\Omega, \mathbf{P})$ , we get to the classical inequality of Rao - Kramer.

**Remark 2.1.** Note that in general case the quantity  $\| l \|_{CLT(Y)}$  dependent on the parameter  $\theta$ .

**Definition 2.1.** The r.i. space  $(Y, \| \cdot \|)$  is said to be *strong normal rearrangement invariant*, briefly, s.n.r.i., write  $Y \in s.n.r.i.$ , if the auxiliary space  $(CLT(Y), \| \cdot \|)$  is equivalent to source space  $(Y, \| \cdot \|)$  on the subspace of the centered variables  $\{\eta\}$  from this space:

$$\forall \{\eta(i), \eta(i) \in Y, \mathbf{E}\eta(i) = 0\} \Rightarrow \left\| \sum_{i=1}^n \eta(j) \right\|_Y \leq K(Y) \sqrt{\sum_{i=1}^n (\|\eta(i)\|_Y)^2} \quad (2.7)$$

for some finite constant  $K(Y)$  depending only on the whole space  $Y$ .

It is clear that for mean zero variable  $\eta$   $\|\eta\|_Y \leq \|\eta\|_{CLT(Y)}$ , therefore in s.n.r.i. spaces both the norm are really (linear) equivalent:

$$\|\eta\|_Y \leq \|\eta\|_{CLT(Y)} \leq K(Y) \|\eta\|_Y.$$

The symbol "n" in the abbreviate of definition 2.1 comes from the word "normal".

**Proposition 2.1.** If in addition to the conditions of theorem 2.1 the space  $Y$  is s.n.r.i., then evidently

$$\sqrt{n} \| \hat{\theta} - \theta_0 \|_{Y'} \geq \frac{1}{K(Y) \| l \|_Y}. \quad (2.8)$$

**Definition 2.2.** The r.i. space  $(Y, \| \cdot \|)$  is said to be *weak normal rearrangement invariant*, briefly, w.n.r.i., write  $Y \in w.n.r.i.$ , if for arbitrary centered r.v.  $\eta$  from this space the norm  $\|\eta\|_{CLT(Y)}$  is finite.

**Proposition 2.2.** If in addition to the conditions of theorem 2.1 the space  $Y$  is w.n.r.i., then

$$\overline{\lim}_{n \rightarrow \infty} \sqrt{n} \| \hat{\theta} - \theta_0 \|_{Y'} > 0. \quad (2.9)$$

Recall, see e.g. [7], chapters 2-4, that the centered (or moreover symmetrically distributed) r.v.  $\eta$  belongs to the Domain of Stable Attraction ( $DSA$ ), iff

$$x \rightarrow \infty \Rightarrow \mathbf{P}(\eta < -x) \sim C_1 x^{-\alpha} L(x), \quad \mathbf{P}(\eta > x) \sim C_2 x^{-\alpha} L(x), \quad (2.10)$$

where  $C_1, C_2 = \text{const} > 0$ ,  $L(x)$  is continuous non - negative slowly varying as  $x \rightarrow \infty$  function,  $\alpha = \text{const} \in (0, 2)$ .

We define also the set  $DNA_\infty$  as a set of all centered random variables  $\{\zeta\}$  from the Domain of Normal Attraction  $DNA$  but with finite variation:  $\text{Var}(\zeta) = \infty$ .

**Proposition 2.3.** If the r.i. space  $(Y, \|\cdot\|)$  is such that

$$Y \cap DSA \neq \emptyset \quad (2.11)$$

or

$$Y \cap DNA_\infty \neq \emptyset, \quad (2.12)$$

then this space  $(Y, \|\cdot\|)$  is not weak normal rearrangement invariant space.

**Proof.** It is sufficient to take in the definition (2.2) the mean zero random variable  $\eta$  from the set

$$\eta \in (Y \cap DSA) \cup (Y \cap DNA_\infty)$$

to ensure that the definition (2.2) is not fulfilled.

### 3 Lebesgue - Riesz norm

We consider in this section the case when at the capacity of the space  $Y$  is the classical LebesgueRiesz space  $L_p = L_p(\Omega, \mathbf{P})$ ,  $p = \text{const} \in [1, \infty)$ . We will denote as usually

$$|\eta|_p = [\mathbf{E}|\eta|^p]^{1/p}; \quad q = p/(p-1), p > 1; \quad q = \infty, p = 1.$$

Define also the so - called  $p$  - Fisher's information  $i_p(\theta)$  :

$$i_p(\theta) = i_p(\eta, \theta) := |l(\eta, \theta)|_p = |\partial \log g(\eta, \theta) / \partial \theta|_p, \quad (3.1)$$

and analogously for the sample

$$I_p(\theta) = I(\vec{\xi}, \theta) := |l(\vec{\xi}, \theta)|_p. \quad (3.1a)$$

which coincides with the classical Fisher's information when  $q = p = 2$ .

The expression for  $i_p(\vec{\xi}, \theta)$  may be rewritten as follows

$$i_p(\eta, \theta) = \int_X |g'_\theta(x, \theta)|^p \cdot g^{1-p}(x, \theta) \mu(dx), \quad (3.1b)$$

and analogously for the quantity  $I_p(\vec{\xi}, \theta)$ .

**Theorem 3.1.**

**A.** The space  $L_p = L_p(\Omega, \mathbf{P})$  is not weak normal rearrangement invariant if  $1 \leq p < 2$ .

**B.** The space  $L_p = L_p(\Omega, \mathbf{P})$  is strong normal rearrangement invariant if  $p \geq 2$ .

**Proof.** The first proposition follows immediately from theorem 2.1, as long as the space  $L_p$ ,  $1 \leq p < 2$  contains the symmetric stable distributed random variable  $\zeta$  with the parameter  $\alpha = (p + 2)/2$ ;  $\alpha \in (0, 2)$  :

$$\mathbf{E}e^{it\zeta} = e^{-|t|^\alpha}, \quad t \in R.$$

So, let now  $p \geq 2$ . The using for us inequality (2.7) is a particular case of the famous Rosenthal's inequality [12]:

$$|\sum_{j=1}^n l_j|_p \leq R(p) \cdot \sqrt{n} \cdot |l|_p, \quad p \geq 2, \quad (3.2)$$

where the Rosenthal's "constant"  $R(p)$  may be estimated as follows:

$$R(p) \leq C \cdot \frac{p}{e \ln p}, \quad C \leq 1.77638 \dots \quad (3.3)$$

see [11].

Therefore, we can accept in (2.7) - (2.8)  $K(L_p) = R(p)$  with estimation (3.3).

We conclude for the regular sample and regular unbiased estimate  $\hat{\theta} = \hat{\theta}_n$  :

**Proposition 3.1.**

$$\sqrt{n} \|\hat{\theta}_n - \theta_0\|_q \geq \frac{1}{R(p) i_p(\theta_0)}, \quad q \in (1, 2), \quad p = q/(q - 1). \quad (3.4)$$

**Remark 3.1.** It follows from the triangle inequality that if  $\{\eta_i\}$  are independent, then

$$I_p(\theta) \leq \sum_{i=1}^n i_p(\eta_i, \theta),$$

but it follows from the Rosenthal's inequality more exact as  $n \gg 1$  estimate

$$I_p(\theta) \leq R(p) \cdot \sqrt{\sum_{i=1}^n i_p^2(\eta_i, \theta)}. \quad (3.5)$$

**Remark 3.2.** The notion of  $p$  - Fisher's information  $i_p(\theta)$  in (3.1) and (3.1a) may be generalized on arbitrary r.i. space  $(Y, \|\cdot\|_Y)$  :

$$i_{(Y)}(\theta) = i_{(Y)}(\eta, \theta) \stackrel{def}{=} \|l(\eta, \theta)\|_Y = \|\partial \log g(\eta, \theta) / \partial \theta\|_Y, \quad (3.6)$$

and analogously for the sample

$$I_{(Y)}(\theta) = I_{(Y)}(\vec{\xi}, \theta) := \|\vec{l}(\vec{\xi}, \theta)\|_Y. \quad (3.6a)$$

Obviously, if the space  $(Y, \|\cdot\|_Y)$  is w.n.r.i., then for independent variables (observations)  $\{\eta(i)\}$

$$I_{CLT(Y)}(\vec{\xi}, \theta) \leq K(Y) \sqrt{\sum_{i=1}^n [i_{(Y)}(\eta(i), \theta)]^2}. \quad (3.7)$$

If in addition the space  $(Y, \|\cdot\|_Y)$  is s.n.r.i., then for independent variables (observations)  $\{\eta(i)\}$

$$I_{(Y)}(\vec{\xi}, \theta) \leq K(Y) \sqrt{\sum_{i=1}^n [i_{(Y)}(\eta(i), \theta)]^2}. \quad (3.8)$$

## 4 Grand Lebesgue space norm

Recently, see [14], [15],[16], [17], [18], [20], [22], [23], [24] etc. appear the so-called Grand Lebesgue Spaces (GLS)

$$G(\psi) = G = G(\psi; (1, B)); \quad B = \text{const} \in (1, \infty]$$

spaces consisting on all the random variables (measurable functions)  $f : \Omega \rightarrow R$  with finite norms

$$\|f\|_{G(\psi)} = \|f\|_{G(\psi; (1, B))} \stackrel{\text{def}}{=} \sup_{p \in (A; B)} \left[ \frac{|f|_p}{\psi(p)} \right]. \quad (4.1)$$

Here  $\psi = \psi(p)$ ,  $p \in [1, B)$  is some continuous positive on the *open* interval  $(1; B)$  function such that

$$\inf_{p \in (A; B)} \psi(p) > 0. \quad (4.2)$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} [1; B)$$

or by abuse of language  $\text{supp}(\psi) = B$ .

The set of all such a functions with the support  $\text{supp}(\psi) = (1; B)$  will be denoted by  $\Psi(1; B) = \Psi(B)$ .

This spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue - Riesz spaces  $L_p$  are extremal case of Grand Lebesgue Spaces, see [24].

Let a function  $\xi : \Omega \rightarrow R$  be such that

$$\exists B > 1 \Rightarrow \forall p \in [1, B) \quad |\xi|_p < \infty.$$

Then the function  $\psi = \psi_\xi(p)$  may be *naturally* defined by the following way:

$$\psi_\xi(p) := |\xi|_p, \quad p \in [1, B). \quad (4.3)$$

More generally, let  $\xi(\alpha)$ ,  $\alpha \in A$ ,  $A$  is arbitrary set, be a *family* of mean zero r.v. such that

$$\exists B > 2, \forall p \in [2, B) \Rightarrow \psi^{(A)}(p) := \sup_{\alpha \in A} |\xi(\alpha)|_p < \infty.$$

The function  $p \rightarrow \psi^{(A)}(p)$  is called a *natural* function for the family  $\xi(\alpha)$ ,  $\alpha \in A$ .

We emphasize that the variables  $\{\xi(\alpha)\}$  can be arbitrarily dependent and that

$$\sup_{\alpha \in A} \|\xi(\alpha)\| G(\psi^{(A)}) = 1.$$

The finiteness of the  $G\psi$  – norm for some r.v.  $\xi$  allows to obtain the exact exponential tail inequalities for the distribution  $\xi$ ; for instance,

$$\sup_{p \geq 1} \left[ \frac{|\xi|_p}{p^{1/m}} \right] < \infty \Leftrightarrow \exists C > 0, \forall x \geq 0 \Rightarrow \mathbf{P}(|\xi| > x) \leq e^{-Cx^m}, \quad m = \text{const} > 0, \quad (4.4)$$

see [22], [24], chapter 1, section 3.

It follows from proposition (2.3) that if  $B < 2$ , then the space  $G(\psi; (1, B))$  is not w.n.r.i. space.

*Therefore, we will suppose in what follows that  $B \geq 2$ ; and we will distinguish two cases:  $2 \leq B < \infty$  and  $B = \infty$ .*

Let us define for arbitrary function  $\psi(\cdot) \in G(\psi; (1, B)) = G(\psi; (2, B))$  the new function

$$\psi_R(p) := R(p) \cdot \psi(p), \quad (4.5)$$

The symbol "R" in (4.5) appears in the honour of Rosenthal.

**Theorem 4.1.** Let  $Y_\psi = G(\psi; (2, B))$ , where  $B > 2$ ; may be  $B = \infty$ . Then  $Y_\psi$  is w.n.r.i. space with

$$CLT(Y_\psi) = G(\psi_R),$$

and

$$\forall \eta : \mathbf{E}\eta = 0, \eta \in Y_\psi \Rightarrow \|\eta\| G(\psi_R) \leq \|\eta\| G(\psi). \quad (4.6)$$

**Proof.** Suppose the mean zero r.v.  $\eta$  belongs to the space  $G(\psi)$ ; we can and will suppose also without loss of generality  $\|\eta\| G(\psi) = 1$ ; then  $|\eta|_p \leq \psi(p)$ ,  $p \in (2, B)$ .

We deduce using Rosenthal's inequality, taking into account the restriction  $p \geq 2$ :

$$|n^{-1/2} \sum_{i=1}^n \eta(i)|_p \leq R(p) \cdot |\eta|_p \leq R(p) \cdot \psi(p) = \psi_R(p),$$

or equally



$$\|n^{-1/2} \sum_{i=1}^n \eta(i)\|G(\psi_R) \leq 1 = \|\eta\|G(\psi),$$

Q.E.D.

Let us define

$$i_{(\psi)}(\theta) = \|l(\eta, \theta)\|G(\psi) = \|\partial \ln g(\eta, \theta)/\partial \theta\|G(\psi),$$

the Fisher's information relative the space  $G(\psi)$ . We conclude on the basis on the conditions of theorem 4.1 in the case of a sample of the volume  $n$  for any regular non - biased estimate  $\hat{\theta}_n$  :

$$\sqrt{n}\|\hat{\theta}_n - \theta_0\|G'(\psi_R) \geq \frac{1}{i_{(\psi)}(\theta_0)}. \quad (4.7)$$

Note that the associate space to the GLS are investigated in the articles [14], [15], [21], [25], [23].

Let us consider the two cases:  $B < \infty$  and  $B = \infty$ .

**First case:**  $B < \infty$ .

In this case holds true the simple estimate:

$$\psi_R(p) \leq K(B) \cdot \psi(p) := C \cdot \frac{B}{e \ln B} \cdot \psi(p), \quad C = 1.7768 \dots,$$

so that the norms  $\|\cdot\|G(\psi)$  and  $\|\cdot\|G(\psi_R)$  and correspondingly the norms  $\|\cdot\|G'(\psi)$  and  $\|\cdot\|G'(\psi_R)$  are equivalent. The inequality (4.7) may be transformed in the considered case as follows:

$$\sqrt{n}\|\hat{\theta}_n - \theta_0\|G'(\psi) \geq \frac{1}{K(B) i_{(\psi)}(\theta_0)}. \quad (4.8)$$

For instance, the function  $\psi(p)$  may be as follows:

$$\psi(p) \asymp (B - p)^{-\gamma} L(B - p), \quad B = \text{const} \geq 2, \quad p \in (2, B),$$

$L(x)$  is positive continuous slowly varying as  $x \rightarrow 0+$  function, see [25], [23].

**Second case:**  $B = \infty$ .

Define the function

$$\psi_m(p) = p^{1/m}, \quad m = \text{const} > 2, \quad p \geq 2,$$

and introduce the following Grand Lebesgue Space  $G_m$  :

$$G_m = G(\psi_m) = \{\eta : \mathbf{E}\eta = 0, \|\eta\|_m := \sup_{p \geq 2} |\eta|_p / p^{1/m} < \infty\}. \quad (4.9)$$

As we know, the *centered* r.v.  $\eta$  belongs to this space iff

$$\mathbf{P}(|\eta| > x) \leq \exp(-(x/C(m))^m), \quad C(m) = \text{const} > 0.$$

Note that the case  $m = 2$  correspondent to the so - called subgaussian variables, see [4], [5], [19], [22], [26].

Let us consider the symmetrical distributed r.v.  $\eta$  such that

$$\mathbf{P}(|\eta| > x) = \exp(-x^m), x > 0;$$

then  $\eta \in G_m$  and  $0 < C_1(m) \leq \|\eta\|_{G_m} \leq C_2(m) < \infty$ ; but evidently

$$\sup_n \|n^{-1/2} \sum_{i=1}^n \eta(i)\|_{G_m} = \infty.$$

This example imply that the space  $CLT(Y)$  may be essentially different from the source r.i. space  $Y$ .

It is easy to verify that in this case the  $CLT(G_m)$  space coincides with the subgaussian space  $G_2$ .

## 5 Exponential Orlicz's norm

Let  $\phi = \phi(\lambda)$ ,  $\lambda \in (-\lambda_0, \lambda_0)$ ,  $\lambda_0 = \text{const} \in (0, \infty]$  be some even strong convex which takes positive values for positive arguments twice continuous differentiable function, such that

$$\phi(0) = 0, \phi''(0) > 0, \lim_{\lambda \rightarrow \lambda_0} \phi(\lambda)/\lambda = \infty. \quad (5.1)$$

We denote the set of all these function as  $\Phi$ ;  $\Phi = \{\phi(\cdot)\}$ .

We say that the *centered* random variable (r.v)  $\xi = \xi(\omega)$  belongs to the space  $B(\phi)$ , if there exists some non-negative constant  $\tau \geq 0$  such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp[\phi(\lambda \tau)]. \quad (5.2).$$

The minimal value  $\tau$  satisfying (4) is called a  $B(\phi)$  norm of the variable  $\xi$ , write

$$\|\xi\|_{B(\phi)} = \inf\{\tau, \tau > 0 : \forall \lambda \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \quad (5.3)$$

This spaces are very convenient for the investigation of the r.v. having an exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem in the Banach space etc.

The space  $B(\phi)$  with respect to the norm  $\|\cdot\|_{B(\phi)}$  and ordinary operations is a Banach space which is isomorphic to the subspace consisting on all the centered variables of *exponential* Orliczs space  $(\Omega, B, \mathbf{P}), N(\cdot)$  with  $N$  - function

$$N(u) = \exp(\phi^*(u)) - 1, \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)). \quad (5.4)$$

The transform  $\phi \rightarrow \phi^*$  is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:

$$\phi^{**} = \phi.$$

The detail investigation of these spaces see in [22], [24], chapters 1,2. For example, this spaces are a particular cases of  $G(\psi)$  spaces. Namely, if  $\lambda_0 = \infty$ , then the  $B(\phi)$  space is isomorphic to the  $G(\psi_\phi)$  space with

$$\psi_\phi(p) = \frac{p}{\phi^{-1}(p)}, \quad p \geq 2,$$

see [22].

There is a proof also in particular that if the mean zero non - trivial r.v.  $\xi$  belongs to the space  $B(\phi)$ , then

$$\max(\mathbf{P}(\xi > x), \mathbf{P}(\xi < -x)) \leq \exp(-\phi^*(x/\tau)), \quad x \geq 0,$$

exponential tail estimate; and is true the inverse inequality: if the centered r.v.  $\xi$  satisfies the last inequality, then it belongs to the space  $B(\phi)$ :  $\|\xi\|B(\phi) \leq C(\phi) \cdot \tau$ .

But we can strengthen in the case of  $B(\phi)$  spaces some assertions on the Grand Lebesgue Spaces.

As in the last section, the function  $\phi(\lambda)$  may be introduced constructively. Namely, let  $\{\xi(\alpha)\}$ ,  $\alpha \in A$  be a family of centered r.v., satisfying the uniform Kramer's condition:

$$\exists C = \text{const} > 0 \Rightarrow \sup_{\alpha \in A} \mathbf{P}(|\xi(\alpha)| > x) \leq \exp(-C \cdot x), \quad x \geq 0.$$

Then we can define

$$\phi^{(A)}(\lambda) := \sup_{\alpha \in A} \ln \mathbf{E} \exp(\lambda \xi(\alpha)), \quad |\lambda| < \lambda_0 = \text{const} > 0.$$

The associate (and dual) space to the Orlicz spaces are described, for example, in the famous book of M.M.Rao and Z.D.Ren [27], chapter 4,5.

Let  $\phi(\cdot) \in \Phi$ ; define a new function

$$\bar{\phi}(\lambda) = \sup_{n=1,2,\dots} [n \cdot \phi(\lambda/\sqrt{n})]; \quad (5.5)$$

then  $\bar{\phi}(\cdot) \in \Phi$ .

For example, if  $\phi(\lambda) \asymp \lambda^Q$ ,  $\lambda \geq 1$ ,  $Q = \text{const} > 1$ , then  $\bar{\phi}(\lambda) \asymp \lambda^{\max(2,Q)}$ ,  $\lambda \geq 1$ .

Notice that if  $Q \geq 2$ , then  $\bar{\phi}(\lambda) \asymp \phi(\lambda)$ ,  $\lambda \geq 1$ . This possibility is absent for the GLS spaces.

The subgaussian r.v. forms the  $B(\phi_2) = B_2$  space with  $\phi_2(\lambda) = 0.5\lambda^2$ ,  $\lambda \in R$ ; in this case  $\bar{\phi}_2(\lambda) = \phi_2(\lambda)$ .

**Theorem 5.1.** Let  $Y_\phi = B(\phi)$ , where  $\phi \in \Phi$ . Then  $Y_\phi$  is w.n.r.i. space with

$$CLT(Y_\phi) = B(\bar{\phi})$$

and

$$\forall \eta : \mathbf{E}\eta = 0, \quad \eta \in Y_\phi \Rightarrow \|\eta\|B(\bar{\phi}) \leq \|\eta\|G(\psi). \quad (5.6)$$

**Proof** is complete analogously to one in theorem 4.1; it based on the equality

$$\mathbf{E} \exp \left( \lambda n^{-1/2} \sum_{i=1}^n \eta(i) \right) \leq \exp \left( \bar{\phi}(\lambda) \right)$$

or equally

$$\|n^{-1/2} \sum_{i=1}^n \eta(i)\|B(\bar{\phi}) \leq \|\eta\|B(\phi). \quad (5.7)$$

Let us define as before

$$i_{(\phi)}(\theta) = \|l(\eta, \theta)\|B(\phi) = \|\partial \ln g(\eta, \theta) / \partial \theta\|G(\phi),$$

the Fisher's information relative the space  $B(\phi)$ . We conclude under the conditions of theorem 5.1 in the case of a sample of the volume  $n$  for any regular non - biased estimate  $\hat{\theta}_n$  :

$$\sqrt{n} \|\hat{\theta}_n - \theta_0\|B'(\bar{\phi}) \geq \frac{1}{i_{(\phi)}(\theta_0)}. \quad (5.8)$$

## 6 Lorentz norm

Recall that the norm of a r.v.  $\zeta$  in the Lorentz space  $L_{p,q} = L_{p,q}(\Omega)$ , more exactly, quasinorm  $\|\zeta\|_{p,q}^*$ ,  $1 \leq p, q \leq \infty$  is defined as follows:

$$\|\zeta\|_{p,q}^* \stackrel{def}{=} \left( \int_0^\infty [\mathbf{P}\{|\zeta| \geq x\}]^{q/p} dx^q \right)^{1/q}, \quad 1 \leq p, q < \infty,$$

and

$$\|\zeta\|_{p,\infty}^* \stackrel{def}{=} \sup_{x>0} \left[ x (\mathbf{P}\{|\zeta| \geq x\})^{1/p} \right].$$

The detail investigation of these spaces see in the books [1], [13]; for instance, it is proved that this quasinorm is linear equivalent to really norm

$$\|\zeta\|_{p,q} := \sup_{A: \mathbf{P}(A)>0} \left[ \frac{\int_A |\zeta(\omega)| \mathbf{P}(d\omega)}{\nu_{p,q}(\mathbf{P}(A))} \right]$$

for some positive for positive values  $z$  function  $\nu_{p,q}(z)$ ,  $z \in (0, 1)$ ;  $\nu_{p,q}(0) = 0$ .

Using for us important facts about these spaces are obtained the book of M.Sh.Braverman [3].

In particular,  $L_{p,p} = L_p$ , therefore the Lorentz are direct generalization of Lebesgue - Riesz spaces. But the exact values of Rosenthal's constants for this spaces are now unknown.

**Theorem 6.1.** Denote  $r = \min(p, q)$ . The Lorentz space  $L_{p,q}(\Omega)$  is strong normal r.i. space iff  $r > 2$  or  $q = 2 \leq p$ .

This assertion is in fact proved in the book of M.Sh.Braverman [3], p. 11 - 13, theorem 7.

## 7 Concluding remarks

The function  $\psi = \psi(p)$  and  $\phi = \phi(\lambda)$  in the sections 4 and 5 may be constructively introduced. Indeed, in the case of  $G(\psi)$  spaces we can introduce the *natural* function for the family of the r.v.  $l(\eta, \theta)$ ,  $\theta \in \Theta$  :

$$\psi_0(p) := \sup_{\theta \in \Theta} \left[ \int_X |g'_\theta(x, \theta)|^p g^{1-p}(x, \theta) \mu(dx) \right]^{1/p}, \quad (7.1)$$

if of course the last expression is finite for some values  $p$  greatest than 2.

Notice that this choice of this function  $\psi_0(p)$  is *optimal*, i.e. minimal.

If for instance the density  $g(x, \theta)$  has a form  $g(x, \theta) = g_0(x - \theta)$ ,  $x, \theta \in R$  ("shift" case), where  $g_0(\cdot)$  is differentiable density function, and  $\mu(A)$  is an ordinary Lebesgue measure, then the integral inside the expression (7.1) does not depended on the value  $\theta$  and hence

$$\psi_0(p) := \left[ \int_R |g'_0(x)|^p g_0^{1-p}(x) dx \right]^{1/p}, \quad (7.2)$$

if of course  $\exists p_0 > 2$ ,  $\psi_0(p_0) < \infty$ .

Another example - scaling parameter. Here again  $X = R$ ,  $\mu$  is Lebesgue measure, but  $\theta \in (\theta_-, \theta_+)$ ,  $\theta_- > 0$  is scaling parameter:

$$g(x, \theta) = \theta^{-1} h(x/\theta),$$

where  $h(\cdot)$  is differentiable density function. Then

$$\left[ \int_X |g'_\theta(x, \theta)|^p g^{1-p}(x, \theta) \mu(dx) \right]^{1/p} = \theta^{-1} \cdot \left[ \int_R |h(y) + y h'(y)|^p \cdot h^{1-p}(y) dy \right]^{1/p}. \quad (7.3)$$

The correspondent natural function  $\phi_0(\lambda)$  for the family  $\{l(\eta, \theta)\}$  in the space  $B(\phi)$  look not so nice as for the Grand Lebesgue Spaces:

$$\phi_0(\lambda) := \sup_{\theta \in \Theta} \log \int_X \exp \left( \lambda \frac{g'_\theta(x, \theta)}{g(x, \theta)} \right) g(x, \theta) \mu(dx), \quad (7.4)$$

if it is finite for some non - trivial interval  $|\lambda| < \lambda_0$ ,  $\lambda_0 = \text{const} > 0$ ; with evident modification for the shift or scale parameter  $\theta$ .

### Conclusions:

The assertions of the sections 2-6 may be simplify as follows: under appropriate conditions on the *weak* r.i. space  $(Y, \|\cdot\|_Y)$ , for example the space  $L_q(\Omega, \mathbf{P})$  with  $1 < q < 2$ , and on the density  $g(\cdot, \cdot)$ , for *arbitrary* unbiased regular sample estimate  $\hat{\theta}_n$  holds true the following inequality:

$$\sqrt{n} \cdot \|\hat{\theta}_n - \theta_0\|_Y \geq C(Y, g(\cdot), \theta_0), \quad n = 1, 2, \dots; \quad (7.4)$$

while for the *MLE* sample estimate  $\tilde{\theta}_n$  and for *strong* r.i. space  $(Z, \|\cdot\|_Z)$ , for example, for  $L_p(\Omega, \mathbf{P})$  with  $p > 2$ ,  $G(\psi)$  and  $B(\phi)$  spaces is true the opposite inequality:

$$\sqrt{n} \cdot \|\tilde{\theta}_n - \theta_0\|_Z \leq \tilde{C}(Z, g(\cdot, \cdot), \theta_0), \quad n = 1, 2, \dots \quad (7.5)$$

## References

- [1] BENNET C, SHARPLEY R. Interpolation of operators. Orlando, Academic Press Inc., (1988).
- [2] BOROVKOV A.A. *Mathematical Statistics*. CRC, (1999), New York, London, Toronto, Hong Kong.
- [3] BRAVERMAN M.SH. *Independent Random Variables and Rearrangement Invariant Spaces*. Cambridge University Press, London Math. Soc., Lecture Notes, series 194, (1994).
- [4] BULDYGIN V.V., KOZACHENKO YU.V. *Metric Characterization of Random Variables and Random Processes*. 1998, Translations of Mathematics Monograph, AMS, v.188.
- [5] [8] Buldygin V.V., Moskvichova K.K. The sub - Gaussian norm of a binary random variable. Theor. Probability and Math. Statist., Kiev, KSU, 2012, 86, p. 33-49.
- [6] IBRAGIMOV I.A., AND KHASMINSKIY R.Z. (1981) *Statistical estimation. Asymptotic theory*. New York - Heidelberg - Berlin; Springer Verlag.
- [7] IBRAGIMOV I.A., AND LINNIK YU.V. (1977). *Independent and Stationary Dependent Random Variables*. Wolters - Noordhoff, Groningen, 1997.
- [8] KOROSTELEV A., KOROSTELEVA O. *Mathematical Statistics. Asymptotic Minimax Theory*. AMS, Providence, Rhode Island, 2011; Graduate Studies in Math.
- [9] LEDOUX M., TALAGRAND M. *Probability in Banach Spaces*. 2002, Springer Verlag, Berlin - Heidelberg - ... - Budapest.
- [10] OSTROVSKY E., ROGOVER E. *Non - asymptotic exponential bounds for MLE deviation under minimal conditions via classical and generic chaining methods*. arXiv:0903.4062v1 [math.PR] 24 Mar 2009.
- [11] OSTROVSKY E. AND SIROTA L. *Schlömilch and Bell series for Bessel's functions, with probabilistic applications*. arXiv:0804.0089v1 [math.CV] 1 Apr 2008.
- [12] ROSENTHAL H.P. *On the Subspaces of  $L_p$  ( $p > 2$ ) spanned by Sequences of independent Variables*. Israel J. Math., 1970, V.3 pp. 273-253.

- [13] E.M.STEIN AND G.WEISS. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Univ. Press, Princeton, N.J., 1987.
- [14] FIORENZA A. *Duality and reflexivity in grand Lebesgue spaces*. Collect. Math. **51**, (2000), 131-148.
- [15] FIORENZA A. AND KARADZHOV G.E. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picone”, Sezione di Napoli, Rapporto tecnico 272/03, (2005).
- [16] IWANIEC T. AND SBORDONE C. *On the integrability of the Jacobian under minimal hypotheses*. Arch. Rat.Mech. Anal., **119**, (1992), 129-143.
- [17] IWANIEC T, KOSKELA P. AND ONNINEN J. *Mapping of Finite Distortion: Monotonicity and Continuity*. Invent. Math. **144**, (2001), 507-531.
- [18] JAWERTH B. AND MILMAN M. *Extrapolation theory with applications*. Mem. Amer. Math. Soc., **440**, (1991), 231-242.
- [19] KAHANE J.P. *Propriétés locales des fonctions à séries de Fourier aléatoires*. Studia Math. (1960), 19, No 1, 1-25.
- [20] KARADZHOV G.E. AND MILMAN M. *Extrapolation theory: new results and applications*. J. Approx. Theory, **113**, (2005), 38-99.
- [21] VAKHTANG KOKILASHVILI, ALEXANDER MESKHI, HUMBERTO RAFEIROE. *Grand Bochner-Lebesgue space and its associate space*. Journal of Functional Analysis, Volume 266, Issue 4, 15 February 2014, Pages 2125-2136.
- [22] KOZACHENKO YU.V., OSTROVSKY E.I. *Banach spaces of random variables of subgaussian type*. Theory Probab. and Math. Stat., Kiev, (1985), p. 42-56 (in Russian).
- [23] LIFLYAND E., OSTROVSKY E., SIROTA L. *Structural Properties of Bilateral Grand Lebesgue Spaces*. Turk. J. Math.; **34**, (2010), 207-219.
- [24] OSTROVSKY E.I. *Exponential Estimations for Random Fields*. Moscow-Obninsk, OINPE, (1999), (in Russian).
- [25] OSTROVSKY E. AND SIROTA L. *Moment Banach spaces: Theory and applications*. HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233-262; (2007), Holon Institute of Technology, Holon city, Israel.
- [26] OSTROVSKY E. AND SIROTA L. *Subgaussian and strictly subgaussian random variables*. arXiv:1406.3933v1 [math.PR] 16 Jun 2014
- [27] RAO M.M., REN Z.D. *Theory of Orlicz spaces*. Marcel Decker; New York, London, Basel, (1991).